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Duality Relation for 32-Vertex Model on the Triangular Lattice

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The duality relation is derived for a vertex model on the triangular lattice. Vertex configurations are limited to the 32 that have an odd number of incoming arrows, and vertex energies are invariant to rotations of $\pi/3$ and reversal of all arrows. Special cases of the model include the triangular Ising model and Baxter's three-spin model, for each of which the duality relation gives the critical temperature.

KEY WORDS: Duality; vertex model; Ising model; three-spin model.

1. INTRODUCTION

A major recent advance in statistical mechanics was achieved with Baxter's solution of the eight-vertex model.[1] The importance of the model rests on several features, but specifically on its considerable generality and on the variety of phase transition behavior exhibited. Special cases of the eight-vertex (or 8-V) model include the dimer, zero-field Ising, "ice," F, and KDP models—all on the two-dimensional square lattice. An unexpected property of the general 8-V model was the continuous dependence of its critical exponents on the energy parameters of the model. The equivalence of the "zero-field" 8-V model and an Ising model with two- and four-body interactions[2]

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has been useful in carrying over to the former techniques of analysis most naturally expressed in terms of spin variables.

Inasmuch as statistical properties of lattice systems can depend qualitatively as well as quantitatively on the geometry of the lattice, with the same dimensionality, it is natural to inquire into the properties of "vertex models" on two-dimensional lattices other than the square lattice.

In general, a vertex model is a lattice system for which each edge has available to it two possible states—for example, the two possible directions of an arrow residing on that edge.[3] The models had their origin in the configurational studies of ice and other hydrogen-bonded solids. The general vertex model for any lattice would permit different energies for each of the 2^c possible configurations at each vertex, c being the coordination number of the lattice. Fundamental simplifications appear if the allowed configurations are restricted to those for which the parity of the number of "incoming arrows" at each vertex is specified. Hence the square lattice eight-vertex model permits only an even number of incoming arrows at each vertex, while the "icelike" models are even more restricted to precisely two incoming arrows. Explicit solution in the 8-V case has been possible only with a different additional restraint, namely that the vertex energies be invariant under reversal of all arrows—the "zero-field" condition.

Wu[4] has studied the general 8-V model on the hexagonal lattice and has shown that its partition function can be transformed to that of the Ising model on the hexagonal lattice, in a magnetic field. Explicit computation of the partition function was possible only when the parity of the number of incoming arrows has one value for one sublattice and the other value for the other sublattice—a condition equivalent to zero magnetic field in the Ising transcription. In the more general case the question of the existence and location of any phase transition was discussed with the help of the Yang–Lee theorem when the equivalent Ising system was ferromagnetic.

Two studies have appeared of vertex models defined on the triangular lattice,[5,6] both restricted to an "icelike" condition of precisely three (i.e., half of the coordination number) incoming arrows at each vertex. Baxter[5] divided the 20 possible vertex energies into three energy classes such that within any one class the configurations are transformed among themselves by arrow reversal or rotations of $\pi/3$. If the vertex weights satisfied a special constraint (i.e., at a special temperature), the "ansatz" approach yielded a dominant eigenvector of the transfer matrix. Kelland[6] showed that a solution may also be obtained (at all temperatures) if the two larger energy classes are assigned the same energy.

In the present paper we generalize the triangular lattice vertex model to allow at each vertex all 32 configurations with an odd number of incoming arrows. It will be seen that invariance to arrow reversal and rotations adds

a fourth energy class (vertices with one or five incoming arrows) to the three introduced by Baxter.[5] Evaluation of the partition function is not attempted, but it is shown that the general problem (with four energy classes) is self-dual and the self-dual temperature is determined using the method of Wegner.[7] It is shown that special cases of the model include the zero-field Ising model and Baxter's three-spin model,[8] both on the triangular lattice.

2. THE MODEL AND ITS ISING EQUIVALENT

Figure 1 shows the 32 possible arrow configurations, already divided into eight classes. Within a class the configurations are transformed among themselves by the combined operations of rotation by $\pi/3$ *and* arrow reversal. There is another way of accomplishing the same division through the "bond" representation[3] suggested also in Fig. 1. Using the single member of class 1 as a "basis," a vertex configuration can be specified by drawing a bond on each edge whose arrow direction differs from that of the basis. The bond description shown at the top of each class of arrow configurations corresponds to the first member of the class. Successive members of each class correspond to all distinct rotated versions of the one bond diagram shown. The further restriction which we shall impose is that the vertex energies be separately

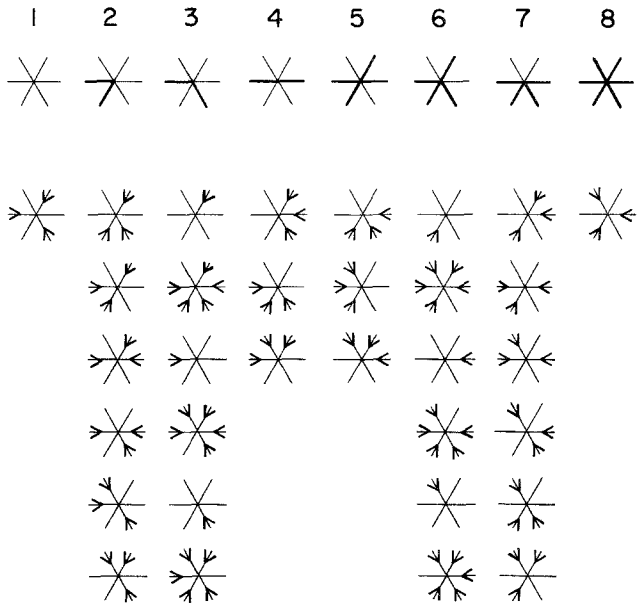

Fig. 1. Vertex configurations. Only the incoming arrows are shown in the lower portion of the figure. The "bond diagrams" at the top of each class of configurations refer to the one configuration of class 1 as basis.

invariant to rotations of $\pi/3$ *or* reversal of all arrows. The effect of this further restriction is the joining of classes j and 9-j of Fig. 1 into one class. There are thus four energy parameters in our model, E_1, E_2, E_3, and E_4.

Just as in the square lattice case,[2] a one-to-two correspondence can be established between the bond configurations of our model and the spin-$\frac{1}{2}$ configurations on the (topological) dual lattice—which in this case is the hexagonal lattice. The correspondence (using the bond description of the vertex model) is that a bond separates unlike spins while an edge without a bond separates like spins. Clearly there are two spin configurations satisfying this rule for any vertex configuration, since there must be an even number of bonds around each vertex. And in turn there is one vertex configuration for every bond "graph" consisting of sets of closed polygons drawn on the triangular lattice.

We now propose the following spin Hamiltonian for the hexagonal Ising spin system:

$$H = -J_0 - J_2 \sum_{2A} \sigma\sigma' - J_2 \sum_{2B} \tau\tau' - J_4 \sum_{4} \sigma\sigma'\tau\tau' - J_4' \sum_{4}{}' \sigma\sigma'\tau\tau' \qquad (1)$$

The spin variables σ and σ' (taking values ± 1) belong to sublattice A and τ and τ' belong to sublattice B. The summations $\sum_{2A}$ and $\sum_{2B}$ are over next-nearest neighbors of the hexagonal lattice (or nearest neighbors of a sublattice), while $\sum_4$ and $\sum_4{}'$ are over sets of four spins contained within any face of the hexagonal lattice. The coupling constant J_4 applies to four successive spins (proceeding around a hexagon) and J_4' applies to four spins at the corner of a rectangle (contained in a hexagon). See Fig. 2. For each hexagonal face of the lattice there are thus six terms in $\sum_4$ and three terms in $\sum_{2A}$, $\sum_{2B}$, and $\sum_4{}'$. It should be noticed that this Hamiltonian is separately invariant to spin reversal on *either* sublattice of the hexagonal lattice.

We find the following correspondence between the vertex energies E_i and the proposed spin coupling constants:

$$E_1 = -J_0 - 6J_2 - 6J_4 - 3J_4', \qquad E_2 = -J_0 - 2J_2 + 2J_4 + J_4'$$
$$E_3 = -J_0 + 2J_2 - 2J_4 + J_4', \qquad E_4 = -J_0 + 2J_2 + 2J_4 - 3J_4' \qquad (2)$$

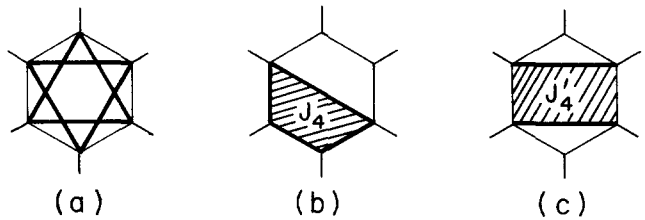

Fig. 2. Interactions within a face of the hexagonal lattice. (a) Each heavy line stands for an interaction J_2. (b) One of six interactions J_4. (c) One of three interactions J_4'.

or conversely

$$J_0 = -\tfrac{1}{16}(E_1 + 6E_2 + 6E_3 + 3E_4), \qquad J_2 = \tfrac{1}{16}(-E_1 - 2E_2 + 2E_3 + E_4)$$
$$J_4 = \tfrac{1}{16}(-E_1 + 2E_2 - 2E_3 + E_4), \qquad J_4' = \tfrac{1}{16}(-E_1 + 2E_2 + 2E_3 - 3E_4) \tag{3}$$

That the spin coupling constants so defined are in fact independent is verified by the nonvanishing of the determinant of coefficients of either (2) or (3).

3. DUAL TRANSFORMATION

General discussions of the dual transformation have been given by Merlini and Gruber[9] and by Wegner.[7] First we notice that the energy h of the six spins around any hexagonal face of the lattice may be expressed in terms of the four "quantum numbers" x_1, x_2, y_1, and y_2 as

$$\begin{aligned} h = {} & -J_2(x_1 + x_2 + x_1x_2 + y_1 + y_2 + y_1y_2) \\ & -J_4(x_1y_2 + x_2y_1 + x_1y_1y_2 + x_2y_1y_2 + x_1y_1y_2 + x_2y_1y_2) \\ & -J_4'(x_1y_1 + x_2y_2 + x_1x_2y_1y_2) \end{aligned} \tag{4}$$

and the total energy H is given by a summation of like expressions over all hexagonal faces. We have here discarded the irrelevant constant J_0 and have defined $x_1,\ldots, y_2$ as in Fig. 3. Each of these four quantum numbers is thus a product of two spins from the same sublattice and takes values ± 1. They may be imagined to reside at the midpoints of the lines joining the pairs of spins whose products define them. These quantum numbers are not, however, all independent variables, as there are constraints governing their possible values.

Wegner[7] has shown how such variables may be employed to effect a dual transformation—i.e., to show the partition function of the original lattice system to be equal (up to an analytic factor) to that of another lattice system. An advantage of Wegner's approach is a direct transformation of the

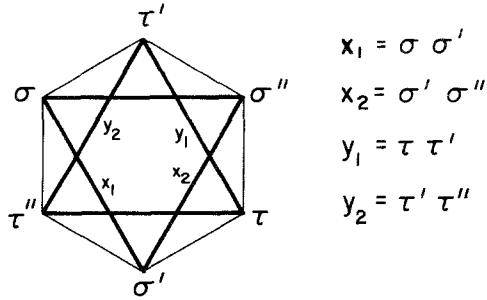

Fig. 3. "Quantum numbers" x_1, x_2, y_1, y_2.

triangular nearest-neighbor Ising lattice to itself (at a different temperature), without the usual intermediate passage through the hexagonal lattice.

The trick for the present problem is to notice that each sublattice of the hexagonal lattice is by itself a triangular lattice. Regarding the spin variables τ on the B sublattice as constant parameters, we first effect Wegner's transformation [Eq. (3.18) of Ref. 7] of the spins σ on the A sublattice. We obtain the result

$$\begin{aligned}
\omega^\dagger(+,+,y) &= \tfrac{1}{2}[\omega(+,+,y) + \omega(+,-,y) + \omega(-,+,y) + \omega(-,-,y)] \\
\omega^\dagger(+,-,y) &= \tfrac{1}{2}[\omega(+,+,y) - \omega(+,-,y) + \omega(-,+,y) - \omega(-,-,y)] \\
\omega^\dagger(-,+,y) &= \tfrac{1}{2}[\omega(+,+,y) + \omega(+,-,y) - \omega(-,+,y) - \omega(-,-,y)] \\
\omega^\dagger(-,-,y) &= \tfrac{1}{2}[\omega(+,+,y) - \omega(+,-,y) - \omega(-,+,y) + \omega(-,-,y)]
\end{aligned} \tag{5}$$

Here $\omega(x_1, x_2, y_1, y_2) = \exp[-\beta h(x_1, x_2, y_1, y_2)]$ is the Boltzmann weight factor at reciprocal temperature β, and y is an abbreviation for (y_1, y_2). The arguments x_1^* and x_2^* of the transformed weights $\omega^\dagger(x_1^*, x_2^*, y_1, y_2)$ are products of pairs of new spins σ^* residing on the B sublattice and the quantum numbers x_i^* may be assigned the same locations as y_i. The weight factors $\omega^\dagger = \omega^\dagger(x_1^*, x_2^*, y_1, y_2)$ for this new lattice system (the "half-transformed" lattice) are defined by Eqs. (5).

We now repeat the process, except this time we keep the spins σ^* fixed and transform the spin variables τ. If Eqs. (5) are expressed by

$$\boldsymbol{\omega}^\dagger(x^*, y) = A_x \boldsymbol{\omega}(x, y) \tag{6}$$

the result of the second transformation may be written as

$$\boldsymbol{\omega}^*(x^*, y^*) = \mathbf{A}_y \omega^\dagger(x^*, y) \tag{7}$$

where $\mathbf{A}_y$ is the same matrix as $\mathbf{A}_x$ but operates on $\omega^\dagger(x^*, y)$ as a function of the y variables with the x^* variables fixed.

Now the new quantum numbers y_1^* and y_2^* are products of pairs of spins τ^* on the A sublattice, and so the y_i^* reside at the original sites of the x_i. Thus the spin system at temperature β has been transformed into itself at a new temperature β^* defined implicitly by

$$\boldsymbol{\omega}^* = \mathbf{A}_y \otimes \mathbf{A}_x \boldsymbol{\omega} \tag{8}$$

where $\mathbf{A}_y \otimes \mathbf{A}_x$ is a 16×16 matrix. Inasmuch as there are only four distinct energy states, however [see Eqs. (2)], we can condense the above equation to

$$\begin{pmatrix} \omega_1^* \\ \omega_2^* \\ \omega_3^* \\ \omega_4^* \end{pmatrix} = \tfrac{1}{4} \begin{pmatrix} 1 & 6 & 6 & 3 \\ 1 & 2 & -2 & -1 \\ 1 & -2 & 2 & -1 \\ 1 & -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} \tag{9}$$

with $\omega_1 = \omega(+, +, +, +), \omega_2 = \omega(+, +, +, -), \omega_3 = \omega(+, -, -, -)$, and $\omega_4 = \omega(+, -, +, -)$, or more briefly by

$$\omega^* = \mathbf{B}\omega \tag{10}$$

where $\boldsymbol{\omega}$ is now a four-component vector as in Eq. (9).

There will be a self-dual temperature, i.e., $\omega^* = \omega$, if **B** has a physically realizable eigenvector corresponding to eigenvalue one. In fact, one is a triply degenerate eigenvalue of **B**, corresponding to the invariant subspace

$$\omega_1 = 2\omega_2 + 2\omega_3 + \omega_4 \tag{11}$$

This equation is our principal result and defines the self-dual temperature when there is one. We now examine some specific cases of the equation.

4. APPLICATIONS

Using Eqs. (2) and $K_2 = 4\beta J_2$, $K_4 = 4\beta J_4$, and $K_4' = 4\beta J_4'$, we may rewrite the self-duality equation as

$$\sinh(K_2 + K_4) = [\exp(-K_4')][\exp(-K_2) + \exp(-K_4)] \tag{12}$$

There are some interesting special cases of this equation. The simplest is $K_4 = K_4' = 0$, under which conditions Eq. (12) reduces to $eK_2 = 3$, the equation for the critical point of the triangular lattice. When the four-body terms vanish, the spin system decouples into two nearest-neighbor Ising lattices as with the spin analog of the square-lattice 8-V model. In the present case, since the decoupled lattices are triangular, there is no transition unless $J_2 > 0$ (ferromagnetic couplings).

Since Eq. (12) is invariant against the exchange of K_2 and K_4, the same self-dual condition $\exp(K_4) = 3$ applies if $K_2 = K_4' = 0$ and the same requirement holds that K_4 be positive for any phase transition.

On the other hand, if $K_2 = K_4 = 0$, there is no finite solution to Eq. (12). This absence of any possibility of critical phenomena can be understood as a consequence of the independence of the contribution of each hexagonal face to the total energy if $J_2 = J_4 = 0$. As indicated in Fig. 4, the entire

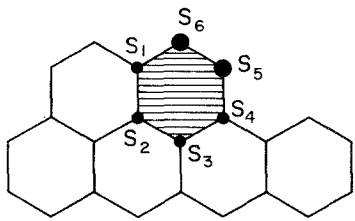

Fig. 4. Special case $K_2 = K_4 = 0$. Each hexagonal face becomes independent.

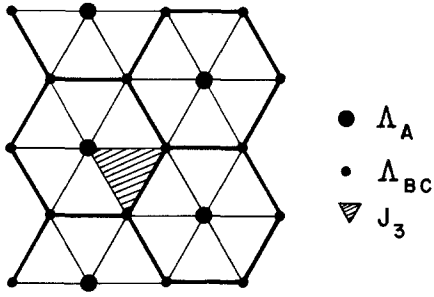

Fig. 5. Partial trace over spins of Λ_A (with three-spin interactions) yields hexagonal lattice Λ_{BC} with interactions as shown in Fig. 2.

lattice may be constructed, one hexagonal face at a time, with the addition of two spins required to complete each face. Labeling those two spins σ_5 and σ_6, and those already assigned values $\sigma_1,\ldots,\sigma_4$, the contribution from that hexagonal face to the energy is

$$\begin{aligned} &-J_4'[\sigma_2\sigma_3\sigma_5\sigma_6 + \sigma_1\sigma_2\sigma_4\sigma_5 + \sigma_1\sigma_3\sigma_4\sigma_6] \\ &\quad = -J_4'\{[1 + t_5\sigma_5][1 + t_6\sigma_6] - 1\} \end{aligned}$$

where $t_5 = \sigma_1\sigma_2\sigma_4$ and $t_6 = \sigma_1\sigma_3\sigma_4$. Clearly the choice $\sigma_i = t_i$ for $i = 5$ and 6 gives energy $-3J_4'$, while the three other choices each gives energy $+J_4'$. Thus the partition function (neglecting edge effects) is simply

$$Z = [3\exp(-\beta J_4') + \exp(+3\beta J_4')]N/2$$

where N is the total number of spins.

The final correspondence we domonstrate is with Baxter's three-spin model on the triangular lattice, where the Hamiltonian is

$$H_3 = -J_3 \sum_3 \sigma\sigma'\sigma'' \tag{13}$$

where $\sum_3$ is over all triangular faces. Now, a finite portion Λ of the triangular lattice may be decomposed as $\Lambda = \Lambda_A \cup \Lambda_{BC}$, Λ_A containing those sites of the A sublattice and Λ_{BC} the rest of the sites. Since the spins σ_A of Λ_A do not interact with each other, it is possible to effect a "partial trace"[10] over the spins σ_A to yield an equivalent system of interacting spins on Λ_{BC}—which is the hexagonal lattice (see Fig. 5).[2] An application of the partial trace technique of Gruber and Merlini[10] quickly shows what many-body interactions exist between the spins of Λ_{BC}.

In the first place the interactions are limited to even-order interactions (since the interactions of the original lattice system were all of the same parity—odd). Second, the interactions are restricted to spins occupying some subset X of some hexagonal face Δ (Fig. 5) of Λ_{BC} (the range of the interactions

[2] An initial use of this approach was attributed by Merlini[11] to F. Wegner.

from the spin σ_A originally at the center of Δ). Finally, for X to correspond to a nonvanishing interaction it must be expressible as a product of an even number of edges D_i of Δ. Here D_i is a set of two neighboring points and "product" means the symmetric difference:

$$D_i D_j = (D_i \cup D_j) - (D_i \cap D_j)$$

Now it is easily learned that the only subsets X meeting these restrictions are precisely those illustrated in Fig. 2. Moreover, using the explicit formulas for the nonvanishing interactions in Ref. 10, it is found that the hexagonal system Λ_{BC} corresponding to the three-spin model has the full symmetry

$$J_2 = J_4 = J_4' = \frac{1}{16}\frac{\cosh 6J_3}{\cosh 2J_3} \tag{14}$$

Finally, transforming back to the vertex representation by way of Eqs. (2), we find the vertex energies equivalent to the three-spin model to be

$$E_2 = E_3 = E_4 = E_1 + 16J_2 \tag{15}$$

Returning to the general 32-V model, with four energy classes, we can make the assertion that the self-duality condition (12) locates the critical temperature of the model—if there is a unique critical point. There is the possibility of multiple transitions, however, and so we withold any such statement about critical temperatures until that possibility has been examined.[12]

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